APPLICATION OF THE LUR'E SYMBOLIC METHOD TO STRESS ANALYSIS AND TO THE TWO-DIMENSIONAL THEORY OF ELASTIC PLATES

(O PRIMENENII SIMVOLICHESKOGO METODA A. I. LUR'E K ANALIZU NAPRIAZHENNYKH SOSTOIANII I DVUMERNYKH TEORII UPRUGIKH PLIT)

PMM Vol.27, No.3, 1963, pp.583-588

U.K. NIGUL
(Tallin)

(Received February 14, 1963)

The state of stress in plates, antisymmetrical about the middle surface, has been studied by means of the symbolic method of Lur'e [1,2]. The possibility of constructing the basic states of stress separately, without taking account of St. Venant edge effects, was analyzed. It was established that boundary value problems exist for which the exact two-dimensional theory [3-8] gives only an illusory refinement of the basic state according to Kirchhoff.

Love [9] considered the construction of a basic state of stress with the following properties: (a) it satisfies exactly all equations of the theory of elasticity; (b) it satisfies exactly the given conditions on the upper and lower surfaces ($z = \pm h$); it has the arbitrariness associated with biharmonic functions; and (d), it differs from Kirchhoff theory by correction terms of the order of a^2 ($a = h/l$, the relative plate thickness). Love [9] determined the basic state of stress separately, with two conditions on each edge, but without analysis of errors.

Lur'e [1,2] developed the theory of the basic state of stress and showed [1] that all other states of stress are rapidly varying. With certain additional conditions they exhibit St. Venant edge effects, localized at the edges and in places where the load or its derivatives has discontinuities.

Lur'e [1], by application of the symbolic method with certain modified solution functions, succeeded in showing that three classes of problems may be distinguished according to the edge conditions, classes for which the state of stress may be determined with different asymptotic errors: class A with error $\vartheta \sim a$, class B with error $\vartheta \sim a^2$, and class C with error $\vartheta \leqslant a^3$.

Clamped plates, cantilevers, plates with edges having zero bending stress (σ_{11}), zero shear stress parallel to the plane of the plate (σ_{12}) and zero normal displacement (u_3), belong to class A. Freely supported plates ($\sigma_{11} = u_2 = u_3 = 0$ at the edge) belong to class B. Freely supported strips and beams ($\sigma_{11} = u_3 = 0$ on the edges) belong to class C, as well as formal problems for which two integral conditions are given initially on the edges and for which the basic state of stress is determined without regard to edge effects, but taking account of displacements or stresses given on the edges.

The separate construction of the basic state of stress in class A or in class B problems does not have the meaning of applying a theory more exact than Kirchhoff's. The approximate calculation with one edge effect (from the three infinite sequences) as considered in two-dimensional theories [3-8] may have an essential significance here only in exceptional cases. Therefore, the exact two-dimensional theories [3-8] do not, generally speaking, guarantee the exact basic state of stress in problems of class A and class B. For class C problems they give less exact results than theories of the separate construction of the basic state of stress [2,9] and the methods of truncated power series [10]. We note that analogous results were obtained not long ago by Gol'denveizer* by the method of asymptotic processes [11].

The difference between the Gol'denveizer method [11] and the asymptotic method of this paper is that the Gol'denveizer method introduces at the very beginning approximations with asymptotic errors of the order of a, a^2, a^3, ..., while the symbolic method of Lur'e consists of an initially exact formal solution which may be expanded in series of a small parameter a. The Kirchhoff theory may be established by such an expansion, as well as formulas for the different approximate methods using truncated power series [10] and the method of asymptotic processes [11,12]. Such an expansion does not appear to be necessary for the numerical solution of problems taking account of edge effects.

1. Basic notation and the initial symbolic formulas. Let E be the modulus of elasticity, μ Poisson's ratio, $2h$ the plate thickness, l a characteristic dimension of the middle surface (in case of sinusoidal loading, no more than a half wavelength), $a = h/l$, the relative plate thickness; ξ, η, ζ dimensionless Cartesian coordinates of which ξ and η

---

* A.L. Gol'denveizer advised the author of this in a conversation held in December of 1962.

are taken to be in the middle surface ($\zeta = 0$) of the plate; u_i ($i = 1, 2, 3$) dimensionless displacements in the ξ, η, ζ directions, respectively; σ_{ij} ($i, j = 1, 2, 3$) dimensionless stresses (multiplied by $(1 + \mu)/E^{-1}$); U_i ($i = 1, 2, 3$) integral displacements, M_{kr} ($k, r = 1, 2$) dimensionless moments, and Q_r, dimensionless transverse forces.

In this notation

$$U_r = \frac{3}{2}\int_{-1}^{+1} u_r \zeta \, d\zeta, \quad U_3 = \frac{1}{2}\int_{-1}^{+1} u_3 \, d\zeta, \quad M_{kr} = \int_{-1}^{+1} \sigma_{kr} \zeta d\zeta, \quad Q_r = \int_{-1}^{+1} \sigma_{r3} \, d\zeta \quad (k, r = 1, 2) \tag{1.1}$$

Following [1] we take as differentiation symbols

$$\frac{\partial}{\partial \xi} = \partial_1, \qquad \frac{\partial}{\partial \eta} = \partial_2, \qquad \partial_1^2 + \partial_2^2 = \Delta, \qquad q = \sqrt{\Delta} \tag{1.2}$$

Assume that antisymmetric loads are given on the upper and lower surfaces, such that

$$\sigma_{r3}(\xi, \eta, \pm 1) = p_r, \qquad \sigma_{33}(\xi, \eta, \pm 1) = \pm p_3, \qquad p_i = p_i(\xi, \eta) \qquad (r = 1, 2; \; i = 1, 2, 3) \tag{1.3}$$

The Lur'e symbolic method permits the construction of the following formulas for the displacements:

$$\begin{aligned} u_r = {} & \partial_r \left[-\frac{q}{2 - 2\mu} (\zeta \sin q \cos q\zeta - \cos q \sin q\zeta) - \sin q \sin q\zeta \right] \varphi_1 - \\ & - 2(-1)^r (\partial_1 + \partial_2 - \partial_r) \frac{\sin q \zeta}{q} \varphi_2 + \\ & + \partial_r \left[-\frac{\zeta}{2 - 2\mu} \cos q \cos q\zeta - \frac{1 - 2\mu}{2 - 2\mu} \cos q \frac{\sin q\zeta}{q} - \frac{1}{2 - 2\mu} \sin q \sin q\zeta \right] \varphi_3 \\ u_3 = {} & \left[\frac{q^2}{2 - 2\mu} (\zeta \sin q \sin q\zeta + \cos q \cos q\zeta) + \frac{1 - 2\mu}{2 - 2\mu} q \sin q \cos q\zeta \right] \varphi_1 + \\ & + \left[-\frac{q}{2 - 2\mu} (\sin q \cos q\zeta - \zeta \cos q \sin q\zeta) + \cos q \cos q\zeta \right] \varphi_3 \end{aligned} \tag{1.4}$$

Expressions for the dimensionless stresses and strains may be obtained as the formulas

$$\sigma_{jj} = \varepsilon_{jj} + \frac{\mu}{1 - 2\mu} \varepsilon, \qquad \sigma_{ij} = \frac{1}{2} \varepsilon_{ij} \qquad (i, j = 1, 2, 3) \tag{1.5}$$

$$\varepsilon_{rr} = \partial_r u_r, \qquad \varepsilon_{33} = \frac{\partial}{\partial \zeta} u_3, \qquad \varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \tag{1.6}$$

$$\varepsilon_{12} = \partial_1 u_2 + \partial_2 u_1, \qquad \varepsilon_{r3} = \partial_r u_3 + \frac{\partial}{\partial \zeta} u_r \qquad (r = 1, 2)$$

Conditions (1.3) are satisfied if solution functions φ_i ($i = 1, 2, 3$) are determined from the equations

$$-D\partial_1\varphi_1 + \cos q\,(\partial_2\varphi_2) = p_1, \qquad -D\varphi_3 = p_3 \qquad \left(D = \frac{\Delta}{2-2\mu}\left[\frac{\sin 2q}{2q} - 1\right]\right) \qquad (1.7)$$
$$-D\partial_2\varphi_1 - \cos q\,(\partial_1\varphi_2) = p_2$$

The trigonometric functions of q or of $q\zeta$ are to be understood as symbolic descriptions of differential expressions of infinite order appearing during their development in the form of power series [1,2]. The differential expressions contain only integral powers of Δ in their expanded form.

2. Elementary states of stress. For brevity, we restrict ourselves to the cases

$$p_s = P_s \cos m\eta, \qquad p_2 = P_2 \sin m\eta, \qquad P_i = P_i(\xi)$$
$$\varphi_s = \Phi_s \cos m\eta, \qquad \varphi_2 = \Phi_2 \sin m\eta, \qquad (s = 1, 3;\ i = 1, 2, 3) \qquad (2.1)$$

where the real parameter m satisfies the condition

$$m \leqslant a. \qquad (2.2)$$

We have, from (2.1)

$$\triangle = \partial_1^2 - m^2 \qquad (2.3)$$

A solution of (1.7) may be constructed [1] in the form

$$\varphi_i = \varphi_i^* + \varphi^\circ_i + \psi_i \qquad (i = 1, 2, 3) \qquad (2.4)$$

Here the φ_i^* denote particular integrals corresponding to the given loadings p_i, and the φ_i° are solutions of the equations

$$\frac{\partial_1\triangle\triangle}{3-3\mu}\varphi_1^\circ + \partial_2\varphi_2^\circ = 0, \qquad \frac{\partial_2\triangle\triangle}{3-3\mu}\varphi_1^\circ - \partial_1\varphi_2^\circ = 0, \qquad \triangle\triangle\varphi_3^\circ = 0 \qquad (2.5)$$

while the ψ_i denote the sums

$$\psi_i = \psi_{i1} + \psi_{i2} + \psi_{i3} + \cdots \qquad (2.6)$$

of special solutions satisfying the conditions

$$\Delta\psi_{sj} = k_j^2\,\psi_{sj}, \qquad \Delta\psi_{2j} = \lambda_j^2\,\psi_{2j} \qquad (s = 1, 3;\ j = 1, 2, \ldots, \infty) \qquad (2.7)$$

in which the k_j are non-zero roots of the equation

$$\sin 2k = 2k \qquad (2.8)$$

and the λ_j are roots of the equation

$$\cos\lambda = 0 \qquad (2.9)$$

The non-zero roots of equation (2.8) are complex and appear in pairs differing in sign. We write the roots with a negative real part in the form

$$k_j = -k_j^{(1)} + ik_j^{(2)}, \qquad k_j = -k_j^{(1)} - ik_j^{(2)} \tag{2.10}$$

Calculation [1] gives for the first values of $k_j^{(1)}$, $k_j^{(2)}$

$$k_1^{(1)} = 3.749,\ k_2^{(1)} = 6.950,\ k_3^{(1)} = 10.119,\ k_1^{(2)} = 1.384,\ k_2^{(2)} = 1.676,\ k_3^{(2)} = 1.858 \tag{2.11}$$

With increasing order of j the magnitudes of $k_j^{(1)}$ and $k_j^{(2)}$ are determined more exactly by the approximate formulas

$$k_j^{(1)} = j\pi + \frac{\pi}{4} - \frac{1}{2}R_j, \qquad k_j^{(2)} = R_j k_j^{(1)}, \qquad R_j = \frac{\ln(4j\pi + \pi)}{2j\pi + {}^1/_2\,\pi} \tag{2.12}$$

Equation (2.9) has the negative roots

$$\lambda_j = -\pi\,(j - {}^1/_2) \tag{2.13}$$

1. The basic state of stress constructed with the aid of φ_i^*, φ_i°. In the special case

$$P_1 = 0, \qquad P_2 = 0, \qquad P_3 = \text{const}, \qquad m = 0 \tag{2.14}$$

one may select

$$\varphi_1^* = 0, \qquad \varphi_2^* = 0, \qquad \varphi_3^* = \frac{1-\mu}{8} P_3\xi^4 \tag{2.15}$$

and from (1.4) formulas are obtained for the displacements

$$u_1^* = P_3\xi\zeta\left[-\frac{1-\mu}{2}\xi^2 - \frac{3\mu}{2} + \frac{2-\mu}{2}\zeta^2\right], \qquad u_2^* = 0 \tag{2.16}$$

$$u_3^* = P_3\left\{\frac{1-\mu}{8}\xi^4 - \frac{3}{4}\xi^2(2-\mu-\mu\zeta^2) + \frac{1}{8}\left[3-\mu+6\,(1-\mu)\,\zeta^2 - (1+\mu)\zeta^4\right]\right\}$$

The scope of this paper does not permit consideration of other cases and so we conclude by referring the reader to [1,2].

For the analysis of the state of stress calculated from the φ_i°, excluding φ_2°, in formulas (1.4) with the aid of the first two equations of the system (2.5), we develop symbolic trigonometric expressions in the form of power series and take $\Delta\Delta\Delta\varphi_1^\circ = 0$, $\Delta\Delta\varphi_3^\circ = 0$. We obtain formulas expressing u_2° in terms of $\Delta\varphi_1^\circ$, φ_3° and their derivatives up to and including the third order. It is shown that the state of stress constructed from $\Delta\varphi_1^\circ$ may be obtained as a linear combination of states of stress corresponding to the function φ_3°. Therefore, it suffices to consider the relation excluding φ_3°. We have as formulas for the displacements

$$u_r^\circ = \left[-1 - \frac{\mu}{2-2\mu}\Delta + \frac{2-\mu}{6-6\mu}\zeta^2\Delta\right]\partial_r \zeta\, \varphi_3^\circ$$
$$u_3^\circ = \left[1 - \frac{2-\mu}{2-2\mu}\Delta + \frac{\mu}{2-2\mu}\zeta^2\Delta\right]\varphi_3^\circ \qquad (2.17)$$

Here, on the basis of (2.1)

$$\varphi_3^\circ = (A_1 e^{m\xi} + A_2 e^{m\xi} + A_3 \xi e^{-m\xi} + A_4 \xi e^{-m\xi}) \cos m\eta \quad \text{for } m \neq 0 \qquad (2.18)$$
$$\varphi_3^\circ = A_1' + A_2'\xi + A_3'\xi^2 + A_4'\xi^3 \quad \text{for } m = 0 \qquad (2.19)$$

The basic state of stress is constructed in the form of sums

$$u_i^1 = u_i^* + u_i^\circ, \qquad \sigma_{ij}^{(1)} = \overset{*}{\sigma}_{ij} + \overset{\circ}{\sigma}_{ij} \qquad (i, j = 1, 2, 3) \qquad (2.20)$$

2. *Edge effects of the St. Venant type.* Assume m to be sufficiently smaller than the values of $k_1^{(1)}$ and $|\lambda_1|$, for example $m \leqslant 1$; then ψ_{ij} is a rapidly decaying (or growing) function of ξ determining edge effects of the St. Venant type. For a sufficiently thin plate, and introducing the coordinate ξ_* inside the plate parallel to the edge, the function ψ_{ij} may be determined from the formulas

$$\psi_{sj} = \Psi_{sj} \cos m\eta, \qquad \psi_{2j} = \Psi_{2j} \sin m\eta \qquad (s = 1, 3;\ j = 1, 2, \ldots, \infty) \qquad (2.21)$$
$$\Psi_{sj} = C_{sj} e^{-\varkappa_j \xi_*} + \bar{C}_{sj} e^{-\bar{\varkappa}_j \xi_*}, \qquad \Psi_{2j} = B_j e^{-\delta_j \xi_*}$$
$$\varkappa_j = +\sqrt{(k_j)^2 + m^2}, \qquad \bar{\varkappa}_j = +\sqrt{(\bar{k}_j)^2 + m^2}, \qquad \delta_j = +\sqrt{\lambda_j^2 + m^2} \qquad (2.22)$$

Here C_{sj}, $\bar{C}_{sj}$ are complex conjugates and B_j is a real constant.

On the basis of (2.21) it is not difficult to obtain formulas from (1.4) for the edge effect displacements, $u_i^{(2)}$. Formulas for the stresses may be obtained from (1.5) and (1.6) and for the integral quantities from (1.1). For example

$$M_{11}^{(2)} = 2m \cos m\eta \sum_{j=1}^{\infty} \left\{\frac{\mu m}{1-\mu}(G_{1j} + \bar{G}_{1j} + G_{3j} + \bar{G}_{3j}) - 2\delta_j \frac{\sin \lambda_j}{\lambda_j^3} \Psi_{2j}\right\}$$
$$Q_1^{(2)} = 2m \cos m\eta \sum_{j=1}^{\infty} \frac{\sin \lambda_j}{\lambda_j} \Psi_{2j} \qquad (2.23)$$

where

$$G_{1j} = \left(\frac{\sin^2 k_j}{k_j^2} - 1\right) C_{1j} e^{-\varkappa_j \xi_*}, \qquad G_{3j} = \frac{\sin^2 k_j}{k_j^2} C_{3j} e^{-\varkappa_j \xi_*} \qquad (2.24)$$

and the formulas for $\bar{G}_{1j}$, $\bar{G}_{3j}$ are obtained from (2.24) by substituting for k_j, κ_j, C_{1j}, C_{3j} the conjugate quantities $\bar{k}_j$, $\bar{\kappa}_j$, $\bar{C}_{1j}$, $\bar{C}_{3j}$. For the case $m = 0$ we have

$$M_{11}^{(2)} \equiv 0, \qquad Q_1^{(2)} \equiv 0 \tag{2.25}$$

but the integral displacements are not zero. Hence, the separate determination of the basic state of stress reduces for $m = 0$ to the statically determinate problem where M_{11} and Q_1 are known on the edges.

The complete state of stress (in the edge effect zones) is determined in the form of the sums

$$u_i = u_i^{(1)} + u_i^{(2)}, \qquad \sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \tag{2.26}$$

3. Example. Limiting asymptotic accuracy of the separate determination of the basic state of stress. Let the load be as in (2.14). On the edges

$$\xi = \pm \xi_0 = \frac{l}{2h} = a^{-1} \gg 1 \tag{3.1}$$

are given the conditions

$$u_s(\xi_0, \zeta) = 0 \qquad (s = 1, 3) \tag{3.2}$$

from which it also follows that

$$U_s(\xi_0) = 0 \qquad (s = 1, 3) \tag{3.3}$$

By virtue of symmetry in the problem we have in (2.19)

$$A_2' = A_4' = 0 \tag{3.4}$$

On the basis of the preceding formulas we have

$$u_1 = u_1^* - 2\xi\zeta A_3' + \sum_{j=1}^{\infty} (K_{1j} + \bar{K}_{1j} + K_{3j} + \bar{K}_{3j})$$

$$u_3 = u_3^* + A_3' \left(\xi^2 - \frac{2-\mu}{1-\mu} + \frac{\mu}{1-\mu}\zeta^2\right) + A_1' + \sum_{j=1}^{\infty} (L_{1j} + \bar{L}_{1j} + L_{3j} + \bar{L}_{3j}) \tag{3.5}$$

$$U_1 = P_3\xi\left[-\frac{1-\mu}{2}\xi^2 + \frac{3}{5}(1-3\mu)\right] - 2\xi A_3' + \sum_{j=1}^{\infty} (S_{1j} + \bar{S}_{1j} + S_{3j} + \bar{S}_{3j})$$

$$U_3 = P_3\left[\frac{1-\mu}{8}\xi^4 - \frac{3-2\mu}{2}\xi^2 + \frac{3-2\mu}{5}\right] + A_1' + A_3'\left[\xi^2 - \frac{6-4\mu}{3-3\mu}\right] + \sum_{j=1}^{\infty} (T_{1j} + \bar{T}_{1j} + T_{3j} + \bar{T}_{3j})$$

Here u_1^* and u_3^* are determined from formulas (2.16), and

$$K_{1j} = k_j C_{1j} e^{ik_j\xi_*}\left[-\frac{k_j}{2-2\mu}(\zeta \sin k_j \cos k_j\zeta - \cos k_j \sin k_j\zeta) - \sin k_j \sin k_j\zeta\right] \tag{3.6}$$

$$K_{3j} = k_j C_{3j} e^{k_j \xi_*} \left[-\frac{\zeta}{2-2\mu} \cos k_j \cos k_j \zeta - \frac{1-2\mu}{2-2\mu} \cos k_j \frac{\sin k_j \zeta}{k_j} - \frac{1}{2-2\mu} \sin k_j \sin k_j \zeta \right]$$

$$L_{1j} = C_{1j} e^{k_j \xi_*} \left[\frac{k_j^2}{2-2\mu} (\zeta \sin k_j \sin k_j \zeta + \cos k_j \cos k_j \zeta) + \frac{1-2\mu}{2-2\mu} k_j \sin k_j \zeta \cos k_j \zeta \right]$$

$$L_{3j} = C_{3j} e^{k_j \xi_*} \left[-\frac{k_j}{2-2\mu} (\sin k_j \cos k_j \zeta - \zeta \cos k_j \sin k_j \zeta) + \cos k_j \cos k_j \zeta \right] \quad (3.6)$$

$$S_{1j} = \frac{3\mu}{1-\mu} k_j C_{1j} e^{k_j \xi_*} \left(\frac{\sin^2 k_j}{k_j^2} - 1 \right), \qquad T_{1j} = C_{1j} e^{k_j \xi_*} \sin^2 k_j$$

$$S_{3j} = \frac{3\mu}{1-\mu} k_j C_{3j} e^{k_j \xi_*} \frac{\sin^2 k_j}{k_j^2}, \quad T_{3j} = C_{3j} e^{k_j \xi_*}$$

The conjugate quantities $\bar{K}_{1j}$, $\bar{K}_{3j}$, $\bar{L}_{1j}$, ... are obtained from (3.6) by substituting for k_j, C_{1j}, C_{3j} the quantities $\bar{k}_j$, $\bar{C}_{1j}$, $\bar{C}_{3j}$. It is easy to prove from (2.8) that the values of K_{3j}, L_{3j}, ... differ from the corresponding values of K_{1j}, L_{1j}, ... only by constant multipliers and so without loss of generality one may set $C_{3j} = \bar{C}_{3j} = 0$.

On the basis of (3.5) and (3.6) it is easy to set up expressions for the displacements on the edges of the plate where $\xi = \pm\, \xi_0$ and $\xi_* = 0$. The condition

$$u_3(\xi_0, \zeta) - U_3(\xi_0) = 0 \quad (3.7)$$

is fulfilled for $\mu \neq 0$ only in the case where C_{1j}, $\bar{C}_{1j}$ exist with values of the order of $P_3 \xi_0^2$. Therefore the separate determination of the coefficients A_1', A_3' – for example in their calculation from conditions (3.3) without taking edge effects into account – is connected with an asymptotic error (as $a \to 0$) of the order of a.

We have the formulas for the basic state of stress

$$u_1^{(1)} = \frac{1-\mu}{2} P_3 \xi_0^2 \, \xi\zeta \left[1 - \frac{\xi^2}{\xi_0^2} + O(a) \right], \qquad \sigma_{13}^{(1)} = -\frac{3}{2} P_3 \xi (1 - \zeta^2) \quad (3.8)$$

$$u_3^{(1)} = \frac{1-\mu}{8} P_3 \xi_0^4 \left[\left(1 - \frac{\xi^2}{\xi_0^2} \right)^2 + O(a) \right], \qquad \sigma_{11}^{(1)} = \frac{1}{2} P_3 \xi_0^2 \zeta \left[1 - \frac{3\xi^2}{\xi_0^2} + O(a) \right]$$

$$M_{11}^{(1)} = M_{11} = \frac{1}{3} P_3 \xi_0^2 \left[1 - \frac{3\xi^2}{\xi_0^2} + O(a) \right], \quad \sigma_{33}^{(1)} = \frac{3}{2} P_3 \zeta \left(1 - \frac{1}{3} \zeta^2 \right)$$

$$Q_1^{(1)} = Q_1 = -2 P_3 \xi$$

Formulas (3.8) – without estimating the errors or the expressions for $\sigma_{13}^{(1)}$, $\sigma_{33}^{(1)}$ – represent in themselves the solution of the Kirchhoff theory. Accuracy of the order of a^2, the proposed refinement of theories [3-8], is useless in the given case. The results and conclusions would be the same if the homogeneous boundary conditions (3.2) were substituted into functions $f_1(\zeta)$, $f_3(\zeta)$, which with their derivatives have an order not greater than $P_3\xi_0{}^2$. It is possible to choose functions $f_1(\zeta)$, $f_3(\zeta)$ which vary rapidly with ζ and do not succeed completely in a separate determination of the state of stress. With an increasing value of j the edge effects vary more rapidly with ζ and the coefficients C_{1j}, $\overline{C}_{1j}$ decrease. In actual problems, therefore, it is sufficient to limit oneself to taking account of a finite number of edge effects (coefficients C_{1j}, $\overline{C}_{1j}$) even though in the asymptotic sense as $a \to 0$ they are all of "identical order".

The solution (3.8) may also be refined by the method of asymptotic processes [11-13], to such a stage that the refinement leads to the construction of edge effects similar to the above.

For example, if the first of conditions (3.2) with the condition $\sigma_{11}(\xi_0, \zeta) = 0$ are substituted, then $u_1{}^{(1)}$ and all stresses in the basic state of stress are determined separately and exactly, and $u_3{}^{(1)}$ to an asymptotic error of the order of a^4. This is connected with the properties (2.25) of edge effects and creates comparatively favorable conditions for the refinement of two-dimensional theories. This is analogous to the situation for statically determinate problems in the calculation of circular plates under uniformly distributed load as considered by Reiss [13]; therefore the result of Reiss does not contradict the conclusions regarding the accuracy of the theory of Reissner.

BIBLIOGRAPHY

1. Lur'e, A.I., K teorii tolstykh plit (On the theory of thick plates). *PMM* Vol. 6, No. 1, 1942.

2. Lur'e, A.I., *Prostranstvennye zadachi teorii uprugosti (Spatial problems of the theory of elasticity)*. Gostekhizdat, 1955.

3. Reissner, E., On the theory of bending of elastic plates. *J. Math. and Phys.*, Vol. 23, 1944.

4. Reissner, E., The effect of transverse shear deformation on the bending of elastic plates. *J. Appl. Mech.*, Vol. 12, No. 1, 1945.

5. Reissner, E., On bending of elastic plates. *Quart. of Appl. Math.*, Vol. 5, No. 1, 1947.

6. Schäfer, M., Über eine Verfeinerung der klassischen Theori dünner schwach gebogener Platten. *Z. angew. Math. und Mech.*, Vol. 32, No. 6, 1952.

7. Ambartsumian, S.A., *Teoriia anizotropnykh obolochek (Theory of anisotropic shells)*. Fizmatgiz, 1961.

8. Gol'denveizer, A.L., O teorii izgiba plastinok Reissnera (On the Reissner theory of plate bending). *Izv. Akad. Nauk, SSSR, otd. tekh. n.*, No. 4, 1958.

9. Love, A.E.H., *A treatise on the mathematical theory of elasticity.* Cambridge University Press, 1934.

10. Kil'chevskii, N.A., Obobshchenie sovremennoi teorii obolochek (Generalization of contemporary shell theory). *PMM* Vol. 2, No. 4, 1939.

11. Gol'denveizer, A.L., Postroenie priblizhennoi teorii izgiba plastinki metodom asimptoticheskogo integrirovaniia uravnenii teorii uprugosti (Construction of an approximate theory of plate bending by the method of asymptotic integration of the equations of the theory of elasticity). *PMM* Vol. 26, No. 4, 1962.

12. Friedrichs, K.O. and Dressler, R.F., A boundary layer theory for elastic plates. *Communications for Pure and Appl. Math.*, Vol. 14, No. 1, 1961.

13. Reiss, E.L., Symmetric bending of thick circular plates. *J. Soc. Industr. Appl. Math.*, Vol. 10, No. 4, 1962.

Translated by E.Z.S.